Beijing
December 2019
9ème Cours

Periods of p-divisible groups and modifications of vector bundles

Recall: k perfect field of car. $p > 0$

$$CW^u \subset CW = \left\{ [x_n]_{n \leq 0} \mid (x_n)_{n \leq N} \text{ is nilpotent for } N \ll 0 \right\}$$

$$CW^u = \left\{ [x_n]_{n \leq 0} \mid x_n = 0 \text{ for } n \ll 0 \right\}$$

Dieudonné Correspondence:

$$\{ p\text{-div. groups}/k \} \xrightarrow{\sim} \{ \text{Dieudonné Crystals} \}$$

$$H \longmapsto \mathrm{Hom}(H, CW)$$

$$\mathrm{Hom}_{W(k)[F,V]}(M, CW) \longleftarrow (M, F, V)$$

$\underline{\text{Ex}}$: $\lambda = \frac{d}{h} \in]0,1]$ $\quad (d,h) = 1,\ h \geq 1$

$$H_\lambda = \ker\left(V^d - F^{h-d} : CW \longrightarrow CW\right)$$

$$= \left\{ [x_i]_{i\leq 0} \in CW \,/\, x_{i-d} = x_i^{p^{h-d}} \right\} \longleftarrow F^{h-d}$$

V^d = shift by d on the right

$$\simeq \mathrm{Spf}\left(k[\![x_0, \ldots, x_{-d+1}]\!]\right)$$

d- dimensional formal p-divisible group height h.

$M_\lambda = \mathbb{D}(H_\lambda)$ $\quad (M_\lambda[\tfrac{1}{p}], F)$ isoclinic simple of slope λ

If $[X_k]_{k\geq 0} + [Y_k]_{k\geq 0} = [P_k(X_0, \ldots, X_k, Y_0, \ldots, Y_k)]_{k\geq 0}$

$P_k \in \mathbb{Z}[X_0, \ldots, X_k, Y_0, \ldots, Y_k]$ gives the addition law of the Witt vectors addition law for the formal group H_x

then

$$(x_0, \dots, x_{-d+1}) \underset{H_x}{+} (y_0, \dots, y_{-d+1})$$

$$= \lim_{k \to +\infty} P_{kd}\left(x_{-d+1}^{p^{k(h-d)}}, \dots, x_0^{p^{k(h-d)}}, \dots, x_{-d+1}, \dots, x_0 ; y_{-d+1}^{p^{k(h-d)}}, \dots, y_0^{p^{k(h-d)}}, \dots, y_{-d+1}, \dots, y_0\right)$$

for the $(x_0, \dots, x_{-d+1}, y_0, \dots, y_{-d+1})$-adic topology (exists thanks to Fontaine)

$$\in \mathbb{F}_p[[x_0, \dots, x_{-d+1}, y_0, \dots, y_{-d+1}]]$$

Period isomorphism in characteristic p

$F \mid \overline{\mathbb{F}}_p$ perfectoid field

H p-divisible formal group $/ \overline{\mathbb{F}}_p$

$M = \mathbb{D}(H)$ Contravariant Dieudonné module

$BW = \varprojlim_{V} CW$ Witt-Bivectors

$= \{ [x_n]_{n \in \mathbb{Z}} \mid (x_n)_{n \leq N}$ is nilpotent for $N \ll 0 \}$

$$0 \longrightarrow W \longrightarrow BW \longrightarrow CW \to 0$$

$$H(O_F) := \mathcal{H}om(\mathrm{Spf}(O_F), H)$$
$$= \varprojlim_{(0) \neq \mathfrak{a} \subset O_F} H(O_F/\mathfrak{a})$$

$$CW(O_F) := \varprojlim_{\mathfrak{a}} CW(O_F/\mathfrak{a})$$

seen as a topological ring

$= \{ [x_n]_{n \leq 0} \mid x_n \in O_F, \liminf_{n \to -\infty} |x_n| < 1 \}$

i.e. $(x_n)_{n \leq N}$ is topologicaly nilpotent for $N \ll 0$.

$$H(\mathcal{O}_F) = \mathrm{Hom}_{W(\mathbb{F}_q)[F,V]}\left(M, CW(\mathcal{O}_F)\right)$$

H formal $\Leftrightarrow$ F top. nilpotent on M

$BW(\mathcal{O}_F) \twoheadrightarrow CW(\mathcal{O}_F)$ induces (use F top. nilp. on M + $\mathcal{O}_F$ perfect)

$$\mathrm{Hom}_{F,V}\left(M, BW(\mathcal{O}_F)\right) \xrightarrow{\sim} \mathrm{Hom}_{F,V}\left(M, CW(\mathcal{O}_F)\right) \ni u$$

with inverse $u \mapsto \left[x \mapsto \lim\limits_{b \to +\infty} F^{-b}\, \widetilde{u(F^b x)} \right]$

in $BW(\mathcal{O}_F)$ for the topology whose basis of neighborhood of 0 is given by $\left(p^b W(\mathcal{O}_F)\right)_{b \geq 0}$

any lift to $BW(\mathcal{O}_F)$ of $u(F^b x) \in CW(\mathcal{O}_F)$

If $(D, \varphi) = (M[\frac{1}{p}], F)$ one deduces

$$\boxed{H(\mathcal{O}_F) = \mathrm{Hom}_{\varphi}\left(D, BW(\mathcal{O}_F)\right)}$$

Now

$$BW(\mathcal{O}_F) = \left\{ \sum_{n\in\mathbb{Z}}' V^n[x_n] \;\middle|\; x_n \in \mathcal{O}_F,\ \liminf_{n\to-\infty} |x_n| < 1 \right\}$$

$$\begin{tikzcd} BW(\mathcal{O}_F) \arrow[d] & \sum_{n\in\mathbb{Z}} V^n[x_n] \arrow[d, mapsto] \\ B_F = \mathcal{O}(Y_F) & \sum_{n\in\mathbb{Z}}' [x_n^{p^{-n}}]\, p^n \end{tikzcd}$$

$$BW(\mathcal{O}_F) = \left\{ \sum_{n\in\mathbb{Z}}' [x_n]\, p^n \;\middle|\; x_n \in \mathcal{O}_F,\ \liminf_{n\to-\infty} |x_n|^{p^{+n}} < 1 \right\}$$

$\cap$ sub $\mathbb{Q}_p$-vector space that contains all periods with slope $\in [0,1]$

$$B^+ = \mathcal{O}(Y_F \cup \{y_{crys}\})$$

<u>Prop:</u> $\mathrm{Hom}_\varphi(D, BW(\mathcal{O}_F)) \xrightarrow{\sim} \mathrm{Hom}_\varphi(D, B)$

<u>Example:</u> $H = H_\lambda,\ \lambda = \frac{d}{h}$

$$H_\lambda(\mathcal{O}_F) = B^{\varphi^h = p^d}$$

$$= BW(\mathcal{O}_F)^{\varphi^h = p^d}$$

$$= \left\{ \sum_{k=0}^{d-1} \sum_{n\in\mathbb{Z}} \left[x_k^{p^{-nh}}\right] p^{nd+k} \;\middle|\; x_0, \dots, x_{d-1} \in \mathfrak{m}_F \right\}$$

Ex. $\lambda = 1$. $\mathcal{F}$ = p-typical formal group law with logarithm $\mathcal{L} = \sum_{n\geq 0} \frac{T^{p^n}}{p^n}$ over $\mathbb{Z}_p$.

usual exp ↓

$E(T) = \exp(\mathcal{L}) \in \mathbb{Z}_p[[T]]$ Artin-Hasse exponential

$E : \mathcal{F} \xrightarrow{\sim} \widehat{\mathbb{G}}_m$ ← multiplicative formal gp. law $X+Y+XY$

$$\begin{tikzcd}
(\mathfrak{m}_F, +_{\mathcal{F}}) \arrow[r, "\sim"] \arrow[d, "\cong"', "E(-)"] & B^{\varphi=p} \\
(\mathfrak{m}_F, +_{\widehat{\mathbb{G}}_m}) \arrow[r, "\sim"] & B^{\varphi=p}
\end{tikzcd}$$

$\varepsilon \longmapsto \sum_{n\in\mathbb{Z}} \left[\varepsilon^{p^{-n}}\right] p^n$ (top map)

$\varepsilon \longmapsto \log([1+\varepsilon])$ (bottom map)

Rem. When $\lambda = \frac{d}{h} \notin [0,1]$ $B^{\varphi^h = p^d}$ has no explicit description: the Banach-Colmez space $B^{\varphi^h = p^d}$ is not representable by a perfectoid space.

$\rightsquigarrow$ diamond.

but by an algebraic space for the pro-étale topology.

Periods in unequal characteristic

$C | \mathbb{Q}_p$ alg. closed. $F = C^\flat$

$H/\mathcal{O}_C$ formal p-divisible group.

We are going to look at $\varprojlim_{\times p} H =$ universal cover of H

We just need its C-points.

Suppose $\mathbb{H}/\overline{\mathbb{F}}_p$ with an identification

$$\mathbb{H} \otimes_{\overline{\mathbb{F}}_p} \mathcal{O}_C/p\mathcal{O}_C = H \otimes_{\mathcal{O}_C} \mathcal{O}_C/p\mathcal{O}_C$$

i.e. we give ourselves an $\mathcal{O}_C$-point of a Rapoport–Zink space.

<u>Prop.</u>: $\varprojlim_{\times p} H(\mathcal{O}_C) \xrightarrow{\sim} \varprojlim_{\times p} H(\mathcal{O}_C/p\mathcal{O}_C)$

Inverse given by sending $(x_n)_{n\geq 0}$, $x_n \in H(\mathcal{O}_C/p\mathcal{O}_C)$

to $\left(\lim_{k\to+\infty} p^k \widetilde{x_{n+k}}\right)_{n\geq 0}$

any lift of x_{n+k} to an element in $H(\mathcal{O}_C)$

via $H(\mathcal{O}_C) = \varprojlim_{i\geq 0} H(\mathcal{O}_C/p^i\mathcal{O}_C)$

$\to$ use that H is p-divisible p^∞-torsion $H_\eta \simeq \mathring{\mathbb{B}}^d_C \circlearrowleft \times p$

contracts everything to 0.

Thus: $\varprojlim_{\times p} H(\mathcal{O}_C) = \varprojlim_{\times p} \mathbb{H}(\mathcal{O}_C/p\,\mathcal{O}_C)$

$\mathcal{O}_C/p\,\mathcal{O}_C = \mathcal{O}_F/\varpi\,\mathcal{O}_F$ with $\varpi^{\#} = p$, $F = C^{\flat}$.

Same argument implies

$$\varprojlim_{\times p} \mathbb{H}(\mathcal{O}_F) \xrightarrow{\sim} \varprojlim_{\times p} \mathbb{H}(\mathcal{O}_F/\varpi\,\mathcal{O}_F)$$

$$\|$$

$\mathbb{H}(\mathcal{O}_F)$ since $\mathbb{H}$ formal $\Rightarrow$ $\mathbb{H}(\mathcal{O}_F) = \mathbb{Q}_p$-Banach space

Thus:

$$\mathbb{H}(\mathcal{O}_F) = \varprojlim_{\times p} H(\mathcal{O}_C).$$

$$\|$$

$\mathrm{Hom}_{\varphi}(D, B_F)$ if (D, φ) = contravariant isocrystal of $\mathbb{H}$

6

Rem: More generally this proves, by testing on any perfectoid $\mathbb{Q}_p$-algebra that

$$\varprojlim_{\times p} \underbrace{H_\eta}_{\simeq \mathring{\mathbb{B}}^d_{\mathbb{Q}_p}} \simeq \underbrace{\mathring{\mathbb{B}}^{d,1/p^\infty}_{\mathbb{Q}_p}}_{\text{pré-perfectoid ball}/\mathbb{Q}_p}$$

$$\mathring{\mathbb{B}}^{d,1/p^\infty}_{\mathbb{Q}_p} = \mathrm{Spf}\left(\mathbb{Z}_p[[x_0^{1/p^\infty}, \dots, x_{d-1}^{1/p^\infty}]]\right)_\eta$$

$\log_H : H_\eta \longrightarrow \mathrm{Lie}\, H \otimes \mathbb{G}_a^{\mathrm{rig}}$ the logarithm of the formal group H_η. $\ker\left(\log_H : H(\mathcal{O}_C) \to \mathrm{Lie}\, H[\frac{1}{p}]\right)$

$$= H(\mathcal{O}_C)[p^\infty]$$

Exact sequence

$$0 \longrightarrow V_p(H) \longrightarrow \varprojlim_{\times p} H(\mathcal{O}_C) \longrightarrow \mathrm{Lie}\, H[\tfrac{1}{p}] \to 0$$

$$(x_n)_{n\geq 0} \longmapsto \log_H(x_0)$$

Deduced by applying $\times p$ to the exact sequence

$$0 \to H(\mathcal{O}_C)[p^\infty] \to H(\mathcal{O}_C) \xrightarrow{\log_H} \mathrm{Lie}\, H[\tfrac{1}{p}] \to 0$$

$$H(\mathcal{O}_C)[p^\infty] \left(\begin{array}{c} H_\eta \\ \big\downarrow {\scriptstyle \log_H} \\ \mathrm{Lie}\, H \otimes \mathcal{O}_\eta^{rig} \end{array} \right.$$ étale cover. Étale torsor under $H(\mathcal{O}_C)[p^\infty]$

Thus:

$$\boxed{0 \to V_p(H) \longrightarrow \mathrm{Hom}_\varphi(D, B_F) \longrightarrow \mathrm{Lie}\, H[\tfrac{1}{p}] \to 0}$$

Rewritten in terms of covariant isocrystals

(D, φ) = covariant Dieudonné module of $\mathbb{H}$

7

$$0 \to V_p(H) \to \left(D \underset{\breve{\mathbb{Q}}_p}{\otimes} B\right)^{\varphi = p} \longrightarrow \mathrm{Lie}\, H[\tfrac{1}{p}] \to 0$$

given by Hodge filtration

$$\mathrm{Fil}\, D_C \subset D_C$$

$$\shortparallel$$

$$\omega_{H^D}[\tfrac{1}{p}]$$

$$D_C / \mathrm{Fil}\, D_C = \mathrm{Lie}\, H[\tfrac{1}{p}]$$

$$D \underset{\breve{\mathbb{Q}}_p}{\otimes} B \xrightarrow{\theta} D_C \twoheadrightarrow D_C/\mathrm{Fil}\, D_C.$$

Ex: $\widehat{\mathbb{G}}_m$ ~> fundamental exact sequence.

Prop: $V_p(H) \to \left(D \otimes_{\breve{\mathbb{Q}}_p} B\right)^{\varphi=p}$

induces an isomorphism

$$V_p(H) \otimes B\left[\tfrac{1}{t}\right]^{\varphi=p} \xrightarrow{\sim} \left(D \otimes_{\breve{\mathbb{Q}}_p} B\left[\tfrac{1}{t}\right]\right)^{\varphi=p}$$

$\to$ Use "Poincaré duality" i.e. the compatibility of $V_p(H) \to \left(D \otimes_{\breve{\mathbb{Q}}_p} B\right)^{\varphi=p}$ with Cartier duality

$$\begin{tikzcd}
V_p(H) \times V_p(H^D) \arrow[r] \arrow[d] & \mathbb{Q}_p(1) = \mathbb{Q}_p \cdot t \arrow[d, phantom, "\cap"] \\
\left(D \otimes_{\breve{\mathbb{Q}}_p} B\right)^{\varphi=p} \times \left(D^* \otimes_{\breve{\mathbb{Q}}_p} B\right)^{\varphi=\mathrm{Id}} \arrow[r, "\langle\,,\,\rangle"] & B^{\varphi=p}
\end{tikzcd}$$

Corollary: $H/\mathcal{O}_C$ with $\mathbb{H} \otimes_{\mathbb{F}_1} \mathcal{O}_C/p\mathcal{O}_C = H \otimes_{\mathcal{O}_C} \mathcal{O}_C/p\mathcal{O}_C$

$(D, \varphi) =$ covariant isoc. of $\mathbb{H}$.

The Hodge filtration $\mathrm{Fil}\, D_C \subset D_C$

induces a modification of vector bundles at ∞

$$\boxed{0 \to V_p(H) \otimes_{\mathbb{Q}_p} \mathcal{O}_X \to \mathcal{E}(D, p^{-1}\varphi) \longrightarrow i_{\infty *} D_C/\mathrm{Fil}\, D_C \to 0}$$

i.e. via $\mathcal{E}(D, p^{-1}\varphi)_\infty \otimes \underbrace{k(\infty)}_{C} = D_C$

$$\mathrm{then}\ \mathcal{E}(D, p^{-1}\varphi) \xrightarrow{u} i_{\infty *} D_C$$

~~$u^{-1}(\mathrm{Fil}$~~ $u^{-1}(i_{\infty *} \mathrm{Fil}\, D_C)$ is a trivial vector bundle.